Interplay of Coulomb interaction and spin-orbit effects in multilevel quantum dots

S. Grap,1 V. Meden,1 and S. Andergassen1,2

1Institut für Theorie der Statistischen Physik, RWTH Aachen University and JARA—Fundamentals of Future Information Technology, 52056 Aachen, Germany
2Faculty of Physics, University of Vienna, Boltzmanngasse 5, 1090 Wien, Austria

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We study electron transport through a multilevel quantum dot with Rashba spin-orbit interaction in the presence of local Coulomb repulsion. We focus on the parameter regime in which the level spacing is larger than the level broadening. Motivated by recent experiments, we compute the level splitting induced by the spin-orbit interaction at finite Zeeman fields, which provides a measure of the renormalized spin-orbit energy. This level splitting is responsible for the suppression of the Kondo ridges at finite $B$ characteristic for the multilevel structure. In addition, the dependence of renormalized $g$ factors on the relative orientation of the applied $B$ field and the spin-orbit direction following two different protocols used in experiments is investigated.

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I. INTRODUCTION

In linear response transport through quantum dots at low temperatures, a twofold Kramers degeneracy leads to the spin Kondo effect in presence of a sufficiently strong local interaction $U$, as compared to the level-lead hybridization $\Gamma$. In the Kondo regime, charge fluctuations of the dot are suppressed and the physics is dominated by spin fluctuations. Varying the level positions by an external gate voltage $V_G$, characteristic conductance plateaus, so-called Kondo ridges, of width $U$ appear around odd (average) electron fillings.1–7 Breaking the twofold Kramers degeneracy by a local Zeeman field of amplitude $B$ destroys the Kondo ridge, and the conductance plateau is split into two Lorentzian resonances of width $\propto \Gamma$ along the $V_G$ axis. In contrast, spin-orbit interaction (SOI), although breaking spin-rotational symmetry by designating a certain (spin) direction, does not destroy the Kondo effect.8–11

In the presence of SOI, spin is no longer a good quantum number but a Kramers doublet remains as time-reversal symmetry is conserved. In multilevel dots with initially (at $B = 0$) well-separated levels, increasing $B$ might lead to energetically degenerate states (level crossings) resulting from different orbitals. If one is a spin-up and one a spin-down state, and the gate voltage is tuned such that an electron fluctuates between these states, one might expect the emergence of a spin Kondo effect at finite magnetic fields.11–14 If the orbital quantum number is conserved in the leads in such systems, additional orbital Kondo effects15 and combinations of spin and orbital Kondo effects16 may appear. Here, we consider a setup where the dot orbital quantum number does not arise in the leads and we thus concentrate on the spin Kondo effect. In contrast to the standard $B = 0$ Kondo effect, the one appearing at finite $B$ is not protected by time-reversal symmetry and can be suppressed in presence of a finite SOI.17

We here study the dot setup sketched in Fig. 1 as a minimal model for a multilevel system. It consists of a tight-binding model with two lattice sites 1 and 2 coupled by the electron hopping of amplitude $t$ and connected to two semi-infinite noninteracting leads via tunnel couplings of strength $t_{\beta,j}$ with $\beta = L, R$ and $j = 1, 2$. The onsite energies of the two levels are given by $\epsilon_{1/2} = V_G \pm \delta$. The Rashba SOI identifies the $z$ direction of the spin space and is modeled as an imaginary electron hopping with spin-dependent sign between the two lattice sites.9,18–21 We choose the parameters in such a way as to deal with two well-separated sets of spin degenerate states, that is, the molecular regime. We here exclusively consider the coupling of a magnetic field to the spin degree of freedom (Zeeman term) and neglect its effect on the orbital motion. The Zeeman field can be decomposed in a parallel and an orthogonal component with respect to the $z$ direction. The local Coulomb interaction (charging energy) is modeled as an onsite $U$ as well as a nearest-neighbor interaction $U'$, and treated within an approximate static functional renormalization group (fRG) approach.22

Motivated by recent experiments on InAs quantum dots in the Kondo regime,23–25 we determine the level splitting induced by the SOI at finite $B$ fields. It provides a measure for the renormalized SOI energy. The dependence of this spin-orbit energy on the relative orientation of the Zeeman field and the spin-orbit direction was measured.23,24,26–31 In the interpretation of the data, Kondo correlations were ignored. We here study how those modify the angular dependence of the level splitting. In addition, we investigate how the two-particle interaction affects the orientation dependence of effective $g$ factors, as extracted experimentally following two distinct protocols. In the first, the gate-voltage dependence of the linear conductance23,24,32 is used to extract $g_{\text{Cond}}$, and in the second bias spectroscopy is employed leading to $g_{\text{Levels}}$.23,26,27,33 We will show that the SOI energy and $g_{\text{Levels}}$ both have an overall amplitude which is renormalized by the two-particle interaction. In addition, the functional dependence on the angle between the SOI and Zeeman field direction is modified. The $g$ factor $g_{\text{Cond}}$ remains unaffected. The renormalization due to the two-particle interaction is seen to be competing with asymmetry effects.

The paper is organized as follows. In the next section, we introduce our minimal multilevel dot model and review the basic concepts of the approximate fRG treatment of the Coulomb interaction. In Sec. III, we discuss our results. We first assess the potential of our model in connection with the fRG to describe the experimentally observed effects in Sec. III A. In Sec. III B, we determine the angular dependence of the...
SOI-induced level splitting at finite Zeeman field for the simplest possible model and provide an intuitive physical picture for the interpretation of the finite-bias spectroscopy. In Sec. III C, we discuss the extraction of effective g factors from the gate-voltage dependence of the linear conductance and the bias spectroscopy as well as their respective angular dependencies. Section III D deals with an extension of the considered setup to noninteracting leads by hopping amplitudes \( t_\beta,j \) with \( \beta = L,R \) and \( j = 1,2 \).

\[ H_Z = B \sum_{\sigma} \sum_{j=1,2} [d^\dagger_{j,\sigma}(\sigma e_j)_{\sigma,\sigma} d_{j,\sigma} \sin \phi + d^\dagger_{j,\sigma}(\sigma e_j)_{\sigma,\sigma} d_{j,\sigma} \cos \phi]. \]

For \( \phi = \pm \pi/2 \), the SOI and the B field are (anti)parallel. In this case, physical quantities are similar to the case of a finite SOI with \( t \) replaced by the effective hopping \( t_{\text{eff}} = \sqrt{t^2 + \alpha^2} \). In particular, the Kondo ridges at finite B are preserved (see Sec. III A for a more detailed discussion). The local Coulomb interaction is included by

\[ H_{\text{int}} = U \sum_{j=1,2} \left( n_{j,\uparrow} - \frac{1}{2} \right) \left( n_{j,\downarrow} - \frac{1}{2} \right) + U' (n_1 - 1)(n_2 - 1) \]

for the local \( U > 0 \) and nearest-neighbor \( U' > 0 \) interactions, respectively, with \( n_{j,\sigma} = d_{j,\sigma}^\dagger d_{j,\sigma} \) and \( n_1 = \sum_\sigma n_{j,\sigma} \). In principle, different local interactions \( U_1,U_2 \) on the two sites can be included, but we focus on the case \( U = U_1 = U_2 \). By subtracting \( 1/2 \) from \( n_{j,\sigma} \) in the definition of \( H_{\text{int}} \), the point \( V_G = 0 \) corresponds to half-filling (even in the presence of Coulomb repulsion) of a symmetric serial dot, which will be the main geometry under consideration in the following.

Finally, the dot Hamiltonian is supplemented by a term describing two semi-infinite noninteracting leads, which we model as one-dimensional tight-binding chains

\[ H_{\text{lead}} = -\tau \sum_{\beta=L,R} \sum_{j=1}^{\infty} \sum_{\sigma} [c^\dagger_{\beta,j+1,\sigma} c_{\beta,j,\sigma} + \text{H.c.}], \]

with lead operators \( c^\dagger_{\beta,j,\sigma} \) and equal bandwidth \( 4\tau \). In the following, we choose \( \tau = 1 \) as the energy unit. We note in passing that it is possible to include Rashba SOI terms in the leads. For our case of vanishing magnetic field in the leads, the effects of such terms enter our calculation only in terms of an effective hopping.\(^9\) The consequences for the Kondo temperature were discussed in Refs. 8, 35, and 36. The dot-lead couplings are given by the tunnel Hamiltonian

\[ H_{\text{coup}} = \sum_{\beta=L,R} \sum_{j=1,2 \sigma} [t_{\beta,j} d^\dagger_{j,\sigma}(\sigma e_j)_{\beta,1,\sigma} + \text{H.c.}], \]

with tunnel barriers set by \( t_{\beta,j} \). For simplicity, we here consider only real \( t_{\beta,j} \) (Ref. 37) in the so-called wide-band limit (see, e.g., Ref. 22) in which the tunnel barriers only enter in combination with the local lead density of states evaluated at the chemical potential. For our setup, we find

\[ \Gamma_{\beta,j} = \pi t^2_{\beta,j} \rho_{\text{leads}} \]
The cutoff free problem is restored. By inserting \( \frac{\partial}{\partial t} G_0^0(i\omega) \) into the generating functional with respect to \( \phi \) one obtains the flow equation for the self-energy, i.e., the Dyson equation

\[
\frac{\partial}{\partial \Lambda} \Sigma^A_{a'\alpha'} = -\frac{1}{2\pi} \sum_{\omega=\pm\Lambda} \sum_{b,b'} e^{i\omega\alpha'} G_{b,b}'(i\omega) G_0^{0,i\omega} (\Gamma_{a'\alpha'b} + \text{c.c.})
\]

where the indices \( a,a',b,b' \) label the quantum numbers \( j,\sigma \). \( \Gamma_{a'\alpha'b} \) is the antisymmetrized two-particle vertex, and the interacting Green’s function \( \Sigma^A \) is determined by the Dyson equation

\[
G^A(i\omega) = \left[ G_0^{-1}(i\omega) - \Sigma^A \right]^{-1}
\]

The initial condition for \( \Lambda_0 \to \infty \) is \( \Sigma^{\Lambda=0} = 0.22 \). In zero-order terms, the two-particle vertex is the bare antisymmetrized interaction and reads as

\[
\Gamma_{a'\alpha'b} = \left( U(1 + \delta_{a'a} - \delta_{a\alpha'}) + U' (\delta_{b'b} + 1 + \delta_{b\alpha'}) \right)
\]

Dynamical contributions to \( \Sigma^A \) are generated only at higher orders. As the latter are important for the conductance at finite temperatures \( T > 0 \), the present approximation scheme is restricted to \( T = 0 \). The correct temperature dependence of the (single-dot) Kondo ridge is only captured if the flow of a frequency-dependent two-particle vertex (leading to a flowing frequency-dependent self-energy) is kept. Within our approximation, the matrix elements \( \Sigma_{a'a''} \) of the self-energy at the end of the flow can be interpreted as interaction-induced renormalizations to the noninteracting model parameters such as the SOI and the onsite energies. Furthermore, entirely new matrix elements will be generated if permitted by symmetry. The full propagator including interaction effects is determined via the Dyson equation (8), from which various observables can be computed. While the renormalized effective single-particle energy levels as a function of the bare parameters will be discussed in detail in Sec. III B, we here concentrate on the linear conductance.
$T = 0$, current-vertex corrections vanish and the Kubo formula for the spin-resolved conductance assumes a generalized Landauer-Büttiker form

$$G_{\sigma,\sigma'} = \frac{e^2}{h} |T_{\sigma,\sigma'}(0)|^2,$$

with the effective transmission $T_{\sigma,\sigma'}(0)$ evaluated at the chemical potential. For the present setup, the transmission is given by the matrix elements of the full propagator leading to

$$G = \sum_{\sigma,\sigma'} T_{\sigma,\sigma'}(0)^2 \sum_{j,j'} |t_{ij}f_{R,j}G_{j,\sigma;j',\sigma'}(0)|^2.$$

III. RESULTS

A. Linear conductance

The linear (and finite-bias) transport characteristics allow us to access the physics of dot setups as information-processing devices. In experiments, large ranges of applied gate voltages and Zeeman fields as well as different orientations of the field can be analyzed. The fRG was shown to capture the effects of the two-particle interaction on the linear conductance. The approximate transformation $\mathcal{f}$RG conserves particle-hole symmetry translating into a rough picture of the linear conductance. The approximate multi- and finite-field conductance for general models as well as for models specifically tailored to describe experimental setups. We here compute the conductance for various parameters of our minimal model to study multilevel dots with sizable SOI as realized in experiments. As using experimental parameters without a well-established microscopic model is difficult, we will choose our parameters in the following in such a way as to facilitate the discussion. As a guide for the magnitude of the single-particle parameters, we use Ref. 39. The noninteracting level structure of the isolated dot is schematically shown in Fig. 2 for $\epsilon_1 = \epsilon_2 = V_G = 0$ and can be utilized to obtain a rough picture of the linear conductance. The approximate fRG conserves particle-hole symmetry translating into a symmetric linear conductance with respect to the gate-voltage transformation $V_G \rightarrow -V_G$. For vanishing Zeeman field $B = 0$, there are two well-separated spin-degenerate energy levels at $\epsilon = \pm t_{\text{eff}}$. When those levels cross the chemical potential of the leads, the Kondo effect will lead to the characteristic conductance plateaus of height $G_{\text{Max}} = 2e^2/h$ (for a totally symmetric serial dot). The width of the plateaus is determined by the local Coulomb interaction. In the presence of a finite field $B$ parallel to the spin-orbit direction, additional spin-degenerate levels occur at $V_G = 0$, $B_z = \pm t_{\text{eff}}$ (dashed lines in Fig. 2), giving rise to Kondo correlations. Similarly to the symmetry in the gate voltage, the conductance is also invariant under $B \rightarrow -B$. This symmetry also holds in the following more general cases. If the SU(2) spin symmetry is broken by SOI [defining the $z$ direction in the spin space, see Eqs. (1) and (2)], this finite-$B$ degeneracy can be lifted via a Zeeman field component perpendicular to the SOI, and the resulting anticrossing (full lines in Fig. 2) suppresses the Kondo effect on an exponential scale. For different coupling strengths to the leads, the maximum conductance on the Kondo plateau is reduced by a factor which depends on the dot parameters. Including additional asymmetries in the onsite energies $\epsilon_1 - \epsilon_2 = 2 \delta \neq 0$ gives rise to finite-field Kondo ridges bent with respect to the $V_G$ axis. Aside from the development of the Kondo effect due to the local Coulomb interaction, the nearest-neighbor interaction renormalizes the position of conductance resonances.

With this simplified multilevel quantum-dot model we can provide a good qualitative description of the various parameter regimes of recent experiments. To exemplify this, in Fig. 3 we show an asymmetrically coupled dot with different onsite energies in a Zeeman field with a perpendicular component to the spin-orbit direction. The results feature a reduced conductance

![Fig. 2](image-url)  
**Fig. 2.** (Color online) Schematic representation of the noninteracting level structure of the isolated dot as relevant in Sec. III B.

![Fig. 3](image-url)  
**Fig. 3.** (Color online) Qualitative reproduction of the conduction region of Fig. 1(c) in Ref. 23. The chosen system parameters are $t = \alpha = 1, \delta = 0.9, \phi = 0.46\pi, U = U' = 1.5, t_{L,1} = -0.5, t_{L,2} = 0.45, t_{R,1} = 0.4, t_{R,2} = -0.3$ (Ref. 45).
and a slanted resonance with a pronounced suppression due to the anticrossing of the levels, both characteristic for the asymmetric setup. For the chosen parameters (see the caption of Fig. 3), the $B$ and $V_C$ dependencies of $G$ strongly resemble the experimental data shown in Fig. 1(c) of Ref. 23.

In the following, we use the minimal multilevel model to compute the gap in the single-particle spectrum (see Fig. 2) for different relative orientations of the SOI and the Zeeman field as recently investigated experimentally. A similar level splitting for small Zeeman fields close to the $B$ close to 0 Kondo ridges allows us to determine the effective $g$ factor $g_{\text{ levels}}$. Alternatively, $g_{\text{ cond}}$ can be extracted from the Coulomb blockade peak splitting of the conductance. For both we will compare our theoretical results to the experimental ones. In order to disentangle the effects under consideration from asymmetry effects (possibly enhanced or suppressed by the interaction), we will focus on a serial double-quantum-dot geometry. Detailed results for the conductance of this model were presented in Ref. 17.

**B. Spin-orbit energy $E_{\text{SOI}}$**

Recent experiments on InAs devices, where SOI is relevant, showed Kondo-type features in the linear conductance at finite Zeeman field. Cotunneling spectroscopy allows us to resolve the dependence of the involved energy levels on the $B$ field and its orientation relative to the SOI direction. This shows that any finite orthogonal component of the Zeeman fields lifts the degeneracy of the two states responsible for the Kondo plateau. For the generic situation, an anticrossing of these two states is observed, varying periodically with the orientation of the Zeeman field. The minimal size of the gap as a function of the magnitude of the Zeeman field defines the energy $E_{\text{SOI}}$ (see Fig. 2). For the theoretical description, we consider the symmetrically coupled serial double-dot geometry at vanishing gate voltage and level splitting, i.e., $\epsilon_1 = \epsilon_2 = 0$, $t_{t,1} = t_{t,2} = t_{\text{coup}}$, and $t_{t,1} = t_{R,1} = 0$ in Fig. 1. This implies $\Gamma_1 = \Gamma_2 = \Gamma$. At any (fixed) cutoff value during the RG flow, the single-particle part of our system [omitting the lead terms of Eqs. (3) and (4)] can be described by the following Hamiltonian in the basis of Eq. (5):

$$h = \begin{pmatrix} B_z & B_z + i B_S & -t + i\alpha & -t - i\beta \\ B_z - i B_S & -B_z & -i\beta & t - i\alpha \\ -t - i\alpha & i\beta & B_z & -B_z \\ i\beta & -t + i\alpha & B_z + i B_S & -B_z \end{pmatrix}$$

with all matrix elements depending on the cutoff $\Lambda$. The initial conditions are given by the bare values of the noninteracting system, with $B_z = B \sin \phi$ and $B_x = B \cos \phi$. The Hamiltonian (matrix) $h$ contains the parameters $B_S$ and $\beta$, which are zero initially but are generated by the two-particle interaction during the RG flow. The new parameter $B_S$ is a Zeeman field perpendicular to both the applied Zeeman field and the SOI direction, with opposite orientation on the two dot sites, while the spin-flip hopping $\beta$ is a Dresselhaus SOI term. The appearance of an effective Zeeman field induced by a finite Coulomb interaction in presence of a broken spin symmetry is discussed for quantum dots with ferromagnetic leads and has been observed recently in systems involving SOI.

We find a nonmonotonic dependence of the effective $B_S = B_S^{\beta=0}$, $\beta = \beta^{\alpha=0}$ for varying the initial value of the SOI (given by $\alpha$) and initial relative orientation of the Zeeman field and the SOI parametrized by the angle $\phi$ as shown in Fig. 4.

At the end of the flow, the effective parameters include the renormalization of the initial values due to the Coulomb interaction. Here, we focus on the angular dependence. For the considered setup, the flow equations assume the convenient analytical form in terms of vectors $\tilde{B}_c = (B_x, B_z, B_S)^T$ and $\tilde{t}_c = (\alpha, \beta, t)^T$:

$$\tilde{B}_c = -\frac{U}{\pi D(\Lambda)} [2(\tilde{t}_c \cdot \tilde{B}_c)\tilde{t}_c + f_+ \tilde{B}_c],$$

$$\tilde{t}_c = -\frac{U'}{\pi D(\Lambda)} [2(\tilde{t}_c \cdot \tilde{B}_c)\tilde{B}_c + f_- \tilde{t}_c],$$

where we introduced

$$f = |\Lambda| + \Gamma, \quad f_\pm = f^2 \pm (|\tilde{B}_c|^2 - |\tilde{t}_c|^2),$$

$$D(\Lambda) = \det(i f - h) = f_+^2 + 2f_+ |\tilde{t}_c|^2 + 4(\tilde{t}_c \cdot \tilde{B}_c)^2.$$  

As a consequence, $B_S$ is generated only for both finite initial $B_x$ and $\alpha$ (compare to Fig. 4). For the hopping $\beta$ to be generated, a nonvanishing $B_z$ is additionally required.

Diagonalizing the Hamiltonian $h$ yields the eigenvalues (the symmetric leads contribute a constant imaginary part which is omitted here, compare Sec. III D)

$$\epsilon = \pm \sqrt{|\tilde{t}_c|^2 + |\tilde{B}_c|^2 \pm 2\sqrt{\tilde{B}_c^2|\tilde{t}_c|^2 - (\tilde{t}_c \cdot \tilde{B}_c)^2}}.$$  

They can be interpreted as the single-particle levels of a corresponding noninteracting system. This interpretation has already been applied successfully to the problem of phase lapses in multilevel quantum dots. Figure 5 shows data for finite gate voltages for which pronounced renormalization effects occur if one of the involved levels crosses the leads’ chemical potential. For this situation, the other energy levels are shifted upwards due to charging effects. This is not observed at $V_G = 0$ as no level crosses the chemical potential.
FIG. 5. (Color online) Bare Zeeman field dependence of the effective single-particle energy spectrum (full lines) as well as of the corresponding noninteracting ones (dashed lines). As a guide to the eye, the chemical potential and the crossing position of the renormalized levels are given as straight black lines. The parameters are $V_0 = 1.25$, $t = 1$, $\alpha = 0.6$, $\phi = 0.25\pi$, $U = U' = 1$, $t_{\text{Coup}} = 0.4$.

but the overall shift of the effective levels with respect to the noninteracting ones apparent in Fig. 5 remains and will be of importance in the following. From the two intermediate levels, we determine the spin-orbit energy

$$E_{\text{SOI}} = \min_B \left( 2 \sqrt{|\vec{B}_x|^2 + |\vec{B}_y|^2 - 2 \sqrt{|\vec{B}_x|^2 |\vec{B}_y|^2 - (\vec{B}_z \cdot \vec{B}_y)|^2} \right).$$

The bare initial value $E_{\text{SOI}} = 2 \alpha \cos \phi = 2 \alpha_{\perp}$ is obtained for $B_{\text{min}}^2 = t^2 + \alpha_{||}^2$, with $\alpha_{||} = \alpha \sin \phi$. Due to the complicated nonlinear structure of the flow equations, an analytic expression of the renormalized spin-orbit energy in terms of the bare Zeeman field can not be obtained. The numerical solution of Eqs. (7) or (9) shows that the renormalized effective parameters acquire a nontrivial angular dependence, as seen in Fig. 6. The results depend only quantitatively on the details of the Coulomb interaction and the interdot hopping $t$, as long as $U^0 / \Gamma$ and $t / \Gamma$ are sufficiently large. We here focus on $U = U'$ and $t = 1$. The renormalized spin-orbit energies are shown as solid lines in Fig. 7 and follow the general form of the bare case calculated above: We find a maximum of the level splitting for the perpendicular orientation $\phi = 0$ and a monotonous decrease to $E_{\text{SOI}} = 0$ when increasing $\phi$ towards the parallel configuration at $\phi = \pi/2$. In a detailed examination for more values of the interaction, the maximum is seen to increase linearly with $U$. The similarity of the interacting and noninteracting curves might lead to the expectation that the maximum is given by twice the renormalized SOI parameter $2 \tilde{\alpha}$, but this is not the case. The dashed lines in Fig. 7 show the renormalized spin-orbit energies divided by the angular dependence of the bare case, i.e., $\cos \phi$. The strong deviations from this bare dependence close to the parallel orientation lead to the important result that the functional dependence of the SOI energy on $\phi$ is strongly affected by the two-particle interaction. This can be traced back to the more pronounced renormalization effects of the bare parameters around the parallel configuration as is seen in Fig. 6, while the parameters are mostly unaffected for $\phi \lesssim 0.5\pi$. This behavior can be intuitively understood in the following way: for the parallel configuration, a sufficiently strong Coulomb interaction gives rise to the finite-$B$ Kondo effect related to the level crossing. With the vanishing Kondo effect in presence of a finite orthogonal $B$-field component relative to the SOI, the Coulomb interaction effects appear suppressed as well. We finally note that our results agree qualitatively with the experimental results of Refs. 23 and 26.

FIG. 6. (Color online) Renormalization-induced $\phi$ dependence of the effective parameters. Here, we compensated for the bare angular dependence of the Zeeman fields by dividing by the corresponding trigonometric function. The initial parameters are $t = 1$, $\alpha = 0.6$, $B = 1.2$, $U = U' = 1$, $t_{\text{Coup}} = 0.4$.

FIG. 7. (Color online) Spin-orbit energy versus orientation of the Zeeman field for several values of the interaction $U = U'$ (full lines). The dashed lines show the spin-orbit energies normalized to the angular dependence of the noninteracting case. The initial conditions are given by $t = 1$, $\alpha = 0.6$, $t_B = t_R = 0.3$.

C. Effective $g$ factors

Experimentally, the $\phi$ dependence of effective $g$ factors is studied as well. In this section, we will model two different protocols used for their extraction. In the first one, following the experiments of Refs. 23, 24, and 32, the Coulomb blockade peak splitting around a $B = 0$ Kondo resonance is extracted from the linear conductance defining $g_{\text{Cond}}$. As the fRG reliably reproduces the linear conductance, this procedure is easily adopted. In the second protocol, bias spectroscopy is used to
measure the level splitting in the vicinity of a Kondo resonance, determining $g_{\text{Levels}}$.

1. Effective $g$ factors: $g_{\text{Cond}}$

As seen in linear conductance measurements for $B > 0$, resonance peaks of maximal height $e^2/h$ and width $\Gamma$ develop out of Kondo plateaus, corresponding to the filling of a dot state. From a linear fit to the splitting of these Coulomb blockade peaks at small to intermediate Zeeman fields, we determine $g_{\text{Cond}}$ by identifying it with the slope of the fit as shown in Fig. 8 [see, e.g., the experiments reported in Ref. 24, Fig. 1(c)]. Due to the presence of the $B = 0$ Kondo ridge, there is an offset before the linear behavior sets in. The full angular dependence of $g_{\text{Cond}}$ in the interacting system is shown in Fig. 9, and for comparison also the noninteracting result is displayed. No significant effects of the two-particle interaction are observed. For all values of the interaction, $g_{\text{Cond}}$ exhibits an $S$-shaped dependence on the relative orientation of the Zeeman field and the SOI with a maximum for the parallel configuration and a finite minimum at $\phi = 0$.

![Figure 8](image)

**FIG. 8.** (Color online) Determination of the $g$ factor $g_{\text{Cond}}$ from the gate-voltage dependence of the Coulomb blockade peaks in the linear conductance. Bare parameters: $t = 1$, $\alpha = 1$, $\phi = 0.5\pi$, $U = U' = 0.5$, $g_{\text{Cond}} = 0.3$. The inset shows a linear fit to the extracted gate-voltage difference $\Delta$.

![Figure 9](image)

**FIG. 9.** (Color online) Zeeman field orientation dependence of $g_{\text{Cond}}$ as extracted from the gate-voltage dependence (see Fig. 8) for different values of the Coulomb interaction $U = U' = 0.0, 0.25, 0.5, 1$ and parameters as in Fig. 8.

2. Effective $g$ factors: $g_{\text{Levels}}$

In order to use the fRG to model the effective $g$ factor $g_{\text{Levels}}$, we will make use of the effective-level interpretation, according to the discussion in the previous section.\(^{46}\) Introducing a finite gate voltage as additional parameter, an explicit analytic form of the flow equation as for the spin-orbit energy [see Eq. (9)] is much more difficult to obtain. Thus, we will solve the general flow equation (7) including a numerical inversion of the matrix on the right-hand side of Eq. (8) and extract the low-field splitting from the eigenvalues of the resulting effective Hamiltonian. As for $g_{\text{Cond}}$, we follow the experimental procedure,\(^{30}\) perform a linear fit to the computed splitting, and identify the slope as $g_{\text{Levels}}$. In principle, we can extract a gate-voltage dependence for this quantity, but here we choose the gate voltage such that the linear conductance around the $B = 0$ Kondo resonances is maximal.

We note that in the single-impurity Anderson model, $g_{\text{Levels}}$ can be related to the magnetic susceptibility.\(^{52}\) The noninteracting Hamiltonian with $\epsilon_1 = \epsilon_2 = V_G$ yields the eigenvalues

$$
\epsilon = V_G \pm \sqrt{B^2 + t^2 + \alpha^2 + 2B\sqrt{t^2 + \alpha^2}}
$$

from which the bare $g_{\text{Levels}}$ is identified:

$$
|\Delta\epsilon^{\pm}| \approx \frac{2}{t_{\text{eff}}} \frac{\sqrt{t^2 + \alpha^2}}{B} = g_{\text{Levels}}B \quad \text{for } B \ll 1.
$$

The angular dependence for the symmetric case at fixed $V_G$ is shown in Fig. 10 for different values of the Coulomb interaction. The general form is again $S$ shaped as for $g_{\text{Cond}}$. In contrast to $g_{\text{Cond}}$, for $g_{\text{Levels}}$, we find similar interaction effects as for $E_{\text{SOI}}$. Again, the renormalization of the maximum value of $g_{\text{Levels}}$ is a prominent effect. Comparing the $\phi$ dependence of the interacting curves with the noninteracting ones, we find deviations which are much less pronounced compared to those

![Figure 10](image)

**FIG. 10.** (Color online) Angular dependence of $g_{\text{Levels}}$ as extracted from the level splitting (solid lines) for the same parameters as in Fig. 8 and different values of the Coulomb interaction $U = U' = 0.0, 0.25, 0.5, 1$. In addition, the angular dependence normalized to the noninteracting case is shown (dashed lines).
of the spin-orbit energy (compare dashed lines of Figs. 10 and 7). Even though the qualitative \( \phi \) dependence is similar to that of \( g_{\text{Cond}} \), the strong renormalization of the amplitude implies that \( g_{\text{Levels}} \neq g_{\text{Cond}} \). This makes it necessary to clearly distinguish between these two quantities as well as other contextually similar definitions using different computation methods or extraction protocols.53–57

D. Asymmetry effects

Experimental setups are more accurately modeled if we consider the more general situation of four (asymmetric) couplings and different onsite potentials as depicted in Fig. 1. Conceptually and computationally, this is easily implemented within the rRG approach, but due to the multitude of parameters an in-depth analysis is beyond the scope of this work. Instead, we will use the experimentally motivated parameters of Fig. 3 and calculate the spin-orbit energy and effective \( g \) factors for this asymmetric setup to illustrate asymmetry effects.

1. Spin-orbit energy \( E_{\text{SOI}} \)

While the calculation of the \( g \) factors remains unaffected by the setup, the protocol for the spin-orbit energy has to be slightly expanded. In the experiments, the parameters were tuned in such a way that the dot was at half-filling to ensure maximum degeneracy. In Sec. III B, this half-filling condition was guaranteed for \( V_G = 0 \) due to particle-hole symmetry. In the considered asymmetric setup and for finite two-particle interaction, this does not hold necessarily and we will fix the gate voltage and Zeeman field amplitude so that on average two electrons occupy the dot and the considered level splitting \( E_{\text{SOI}} \) is minimal. For the noninteracting case, the dot is still half-filled at \( V_G = 0 \) and the spin-orbit energy can be computed after diagonalizing Eq. (6) for \( \omega = 0^\circ \). As \( \Gamma_1 \neq \Gamma_2 \) and \( \gamma \neq 0 \), the lead self-energy contribution affects the real (and the imaginary) part of the eigenvalues of the dot system. In Sec. III B, the lead self-energy contribution was proportional to unity and thus the real parts of the eigenenergies of the isolated dot remained unaffected. This renormalization by the leads has a drastic effect on the spin-orbit energy as the dashed lines in Fig. 11 show. The overall structure remains similar to the serial symmetric case of Sec. III B with a reduced maximum (compared to \( 2 \alpha \) in Sec. III B) at \( \phi = 0 \) and a decrease towards larger \( \phi \). The spin-orbit energy then tends towards 0 for \( \phi \approx 0.45\pi \) and remains very small up until \( \phi = 0.5\pi \). The picture for the interacting dot is similar (full line in Fig. 11), but the two-particle interaction decreases the range for which the spin-orbit energy is small compared to the \( U = 0 \) case. In the theory plots, this lead renormalization effect occurs most visibly for large \( \phi \) where the spin-orbit energy is small. We refrain from comparing this effect to the experiment as it might be masked by finite-temperature effects or the resolution in the bias spectroscopy. It is obvious that this effect depends strongly on the detailed geometry of the quantum dot. A more thorough analysis is needed to determine the importance of this result for experiments.

2. Effective \( g \) factors

For \( g_{\text{Cond}} \) we find several interesting features compared to the symmetric case in the previous section. The most obvious effect seen in Fig. 12 is a different strength of the effective \( g \) factor depending on which \( B = 0 \) level degeneracy (\( U = 0 \)) or Kondo resonance (\( U \neq 0 \)) is selected for measurement (\( V_G \leq 0 \equiv g^\pm \)). This is already seen in the noninteracting curves depicted as dashed lines in Fig. 12 and is only weakly affected by the interaction (full lines). An interesting effect is observed if one considers the angular dependence of the noninteracting case. Only \( g^+_{\text{Cond}} \) follows the general form found in Sec. III C, while \( g^-_{\text{Cond}} \) remains nearly constant over the whole \( \phi \) range (slight deviations might be attributed to numerics). If the interaction is turned on, this atypical behavior is not found and both curves follow the general form with only slightly renormalized amplitude compared to the free case.

Computing \( g_{\text{Levels}} \) we find that the angular dependence found in Sec. III C is qualitatively preserved (full lines in Fig. 13). The two-particle interaction renormalization effect on the detailed functional form of the angular dependence is suppressed in comparison to the symmetric setup (dashed lines in Fig. 13). As for \( g_{\text{Cond}} \), a strong dependence on the

![FIG. 11. (Color online) Angular dependence of the spin-orbit energy for the parameters of Fig. 3 (full lines). For reference, the angular dependence of the noninteracting case is shown (dashed lines). The inset shows a zoom-in of the \( \phi > 0.4\pi \) range (note the logarithmic scale on the y axis).](image)

![FIG. 12. (Color online) Angular dependence of \( g^\pm_{\text{Cond}} \) for the parameters of Fig. 3 (full lines). For reference, the dependence of the noninteracting case is shown by the dashed lines.](image)
asymmetry is seen if we consider both $B = 0$ Kondo ridges, but only in the interacting case. As the two sets of states involved in forming the Kondo effect are coupled differently to the leads, the renormalization of parameters close to the resonance is different as well. This results in different $g_{\text{Levels}}$ for the Kondo plateaus at positive and negative gate voltages. This asymmetry effect present in $g_{\text{Cond}}$ as well as $g_{\text{Levels}}$ has already been observed in experiments $^{23,32,33}$ where the two Kondo ridges can be attributed to different orbitals of the device.

**IV. CONCLUSION**

We studied the influence of the Coulomb interaction on the level splitting induced by the spin-orbit interaction (the spin-orbit energy), and the effective $g$ factors in multilevel quantum dots with SOI. Furthermore, we interpret fRG results in terms of effective single-particle energy levels to obtain an intuitive physical picture for the understanding of finite-bias spectroscopy. $^{46}$ For a basic symmetrically coupled serial model, we find that two experimentally investigated quantities are affected by the local Coulomb interaction. In the case of the spin-orbit energy $E_{\text{SOI}}$, the effect on the overall amplitude as well as the dependence on the relative orientation between the SOI and the applied Zeeman field is very pronounced even for intermediate interaction strengths. For the $g_{\text{Levels}}$ factor calculated from the gate-voltage-dependent effective level splitting, which mimics the experimental $g$-factor extraction from bias spectroscopy, we find sizable renormalization of its magnitude, while the angular dependence is only mildly affected. On the other hand, the $g_{\text{Cond}}$ factor extracted from the Coulomb blockade peak splitting of the linear conductance appears to be almost interaction independent. The considered asymmetric parameter set shows qualitatively similar results. They nevertheless reveal a complex interplay of lead coupling, two-particle interaction, and SOI. The presented results are of importance for the understanding of transport measurements of multilevel quantum dots with SOI in presence of an external Zeeman field as reported in Refs. $^{23, 24, 26, 32}$. While the qualitative behavior of the considered quantities is consistent with the experimental data, the observed deviations and interaction-dependent amplitudes in the theoretical calculations provide new directions to investigate in future experiments.

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The more general situation of complex coupling parameters is realized if a magnetic flux pierces the dot geometry. In the case considered here, the couplings are related to the overlap of the dot and lead wave functions, which can be positive or negative.


Extracting the tunnel couplings from experimental data is difficult in general and was not attempted in Ref. 23. Fitting Lorentzians to the Coulomb blockade peaks usually gives reasonable estimates for the magnitude but not for the signs or phases of the tunnel couplings. Additional complications arise if one is interested in a strongly correlated region where the conductance resonances are no longer Lorentzians. We therefore assume tunnel couplings of the order of magnitude but not for the signs or phases of the tunnel couplings. In the analysis of the cotunneling experiments, nonequilibrium effects are often ignored. Our calculations yield the spectrum of the dot system without nonequilibrium effects.


